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Performance evaluation for energy efficient topologic control in ad hoc wireless networks☆

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Abstract

Minimizing total energy to keep an ad hoc wireless network symmetrically connected is an NP-hard problem. Recently, several greedy approximations have been proposed, based on k-restricted decompositions of the network. Their performance ratios are established through estimations of the least upper bound ρ_k for the ratio between total powers of best possible k-restricted decomposition and the optimal solution. In this paper, we determine the exact value of ρ_k for all k.

Keywords: Min power symmetric connectivity; Ad hoc wireless network; Approximation algorithms

1. Introduction

An ad hoc wireless network consists of mobile nodes connected by wireless links. It has no fixed infrastructure and maintains a dynamic topology. To keep symmetric connections in an ad hoc network, any two nodes shall set up a point-to-point wireless connection if the power of each node is large enough to include the other one within its transmission range. This range is a disk centered at the node and with radius r determined by formula $P = cr^{\alpha}$, where P is the power at the node and c and $\alpha \in [2, 6]$ are constants.

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In ad hoc wireless networks, mobile nodes usually use batteries, so their powers are limited. This constraint promoted many efforts on energy efficient routing designs. One of the research issues is to minimize the total energy for keeping symmetric connectivity. This problem has a mathematical formulation as follows [1–3]:

Given a set V of n points in the Euclidean plane, find a spanning tree T to minimize

$$P(T) = \sum_{u \in V} \max_{(uv) \in T} d(u,v)^{\alpha},$$

where $d(u,v)$ is the Euclidean distance.

This problem has been proved to be NP-hard [4] and minimum spanning tree has been shown to have performance ratio exactly two. To obtain better approximation, Călinescu et al., [3] employed the technique of k-restricted decomposition. A k-restricted decomposition is a partition of a spanning tree into small trees each with at most k nodes. There are various greedy algorithms to choose a k-restricted decomposition for constructing approximation [3]. Their performance ratios are established through estimation of least upper bound ρ_k for the ratio between the total powers of best possible k-restricted decomposition and the optimal solution for the same input set of nodes. Călinescu et al., showed that $\rho_k \leqslant 1 + 1/\log k$ for all $k \geqslant 3$ and $\rho_3 \leqslant \frac{7}{4}$. Althaus et al. [1] showed $\rho_3 \leqslant \frac{5}{3}$. In this paper, we show that for any $k \geqslant 3$

$$\rho_k = \frac{(r+1)2^r + s}{r2^r + s},$$

where $k = 2^r + s, 0 \leqslant s < 2^r$.

The rest of the paper is organized as follows:

In Section 2, we introduce some preliminary knowledge and definitions about this problem, and also provide the basic model we need to prove the bounds; Upper bound of ρ_k is proved in Section 3 while Section 4 deals with the lower bound of ρ_k.

2. Preliminaries

Let $G = (V, E, c)$ be an edge-weighted graph. Without loss of generality, we can assume that all edge weights are different. For a connective component G' of G, denote by $C(G')$ the total weight of the edges in G'.

Definition 1. Let $T = (V, F)$ be a spanning tree of some edge-weighted graph G. Define the power-cost of a vertex $v \in V$ with respect to T by

$$p_T(v) = \max_{(uv) \in F} c(uv).$$

Define the power-cost of the tree T by

$$P(T) = \sum_{v \in V} p_T(v).$$

A k-restricted decomposition Q of T is a partition of T into a series of subtrees $\{T_i = (V_i, F_i) | i = 1 \dots p\}$ satisfying

$$|V_i| \leqslant k,$$
$$\bigcup_{i=1}^{p} F_i = F,$$
$$F_i \cap F_j = \emptyset \ (\forall i \neq j).$$

The power-cost of Q is defined by

$$P(Q) = \sum_i P(T_i).$$

For an arbitrary tree T, the minimum power-cost of k-restricted decompositions of T is

$$\min_Q P(Q) \text{ where } Q \text{ decomposes } T$$

Definition 2. For an integer $k \geqslant 1$, denote by ρ_k the supremum, over all trees, of the ratio of the minimum power-cost of k-restricted decompositions of T to the power-cost of T:

$$\rho_k = \sup_T \min_Q \frac{P(Q)}{P(T)}.$$

In order to estimate the upper bound and lower bound for ρ_k, it is necessary to convert a tree $T = (V, F, c)$ to a so-called *binary edge-tree* $B_T = (F, E_F)$ by the following operation (motivated by [3]):

1. Find the max weighted edge $h = (r_1 r_2)$ of T. Notice that removal of h will decompose T into subtrees T_1 and T_2 which are rooted at r_1 and r_2, respectively.

2. For an arbitrary vertex $v \in T_i$, except r_1 and r_2, the edge connecting v to its unique parent is called a *parent edge* of v. (For r_1 and r_2, the edge $h = (r_1 r_2)$ is defined to be their parent edge.) All the other edges incident to v are called *child edges* which can be sorted by their costs in the increasing order. For the heaviest edge e in v's child edges, we define *next*(e) as v's parent edge. For some other child edge e, we define *next*(e) as the adjacent heavier edge in the increasing order above.

3. Establish $B_T = (F, E_F)$ whose vertex set is F and edge set is $E_F = \{(e, next(e)) | e \in F\}$. B_T is a binary tree since for all $e \in F$ there exist at most two edges e_i $(i = 1, 2)$ such that *next*$(e_i) = e$.

Without loss of generality, we can assume that B_T is a complete binary tree by adding virtual nodes and virtual zero-weighted edges to T.

It is obvious that B_T is a vertex-weighted tree and a connective component T' of T corresponds to a connective component $B_{T'}$ in B_T. We also denote by $C(B_{T'})$ the total weights in $B_{T'}$ without ambiguity.

As shown in Fig. 1, a k-restricted decomposition $Q = \{T_1, T_2, \dots, T_p\}$ in T corresponds to a $(k-1)$-*restricted vertex decomposition* R in B_T. More formally, we give

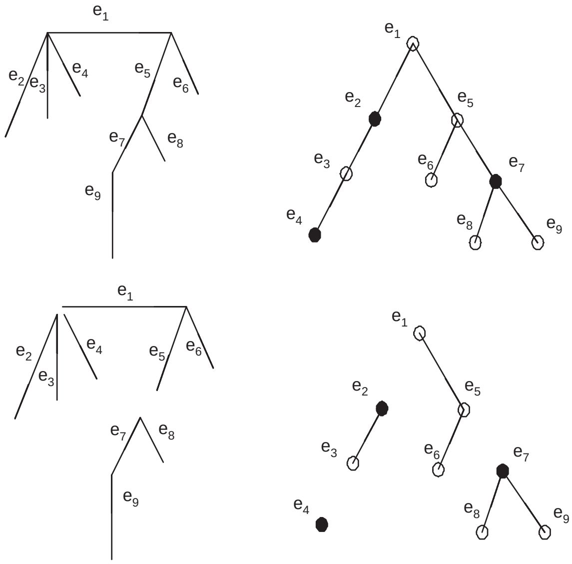

Fig. 1. The correspondence between a tree and its edge-tree.

the explicit definition: a $(k-1)$-*restricted vertex decomposition* R of $B_T = (F, E_F)$ is a partition of B_T into a series of subtrees $\{B_i = (F_i, E_i) | i = 1 \dots p\}$ satisfying that

$$|F_i| \leqslant k-1,$$
$$\bigcup_{i=1}^{p} F_i = F,$$
$$F_i \cap F_j = \emptyset \ (\forall i \neq j).$$

In the rest of the article, we always take the root of B_1 as the heaviest edge h in T. Without ambiguity, a $(k-1)$-restricted vertex decomposition R of some binary edge-tree B is also called a $(k-1)$-restricted decomposition of B.

Given a pair of decomposition $Q = \{T_1, T_2, \dots, T_p\}$ and $R = \{B_1, B_2, \dots, B_p\}$, if we denote by e_i the root of B_i, the increase of the power-cost can be expressed as

$$P(Q) - P(T) = \sum_{i=2}^{p} \min\{c(e_i), c(next(e_i))\}. \tag{1}$$

For a subtree B_i in R, its contribution to the increase of the power-cost can be bounded by the weights of its cutoff children. From this heuristic observation, a notation is introduced

Definition 3. Let $D(B_i)$ denote the set

$$\{e \in F \mid \mathrm{next}(e) \in B_i \text{ while } e \notin B_i\},$$

then the power-cost contribution $I(B_i)$ is defined to be

$$I(B_i) = \sum_{e \in D(B_i)} c(e). \tag{2}$$

The power-cost contribution of the decomposition R is defined by

$$I(R) = \sum_{i=1}^{p} I(B_i). \tag{3}$$

The following proposition holds naturally from the definition above.

Proposition 4. *If a $(k-1)$-restricted decomposition $R = \{B_1, B_2, \ldots, B_p\}$ of B_T corresponds to a k-restricted decomposition Q of T, then*

$$P(Q) \leqslant P(T) + I(R) \tag{4}$$

and the equality holds if and only if $c(e_i) < c(next(e_i))$ for all $2 \leqslant i \leqslant p$ where e_i is the root of B_i.

3. Upper bound for ρ_k

The proof of the upper bound for the k-restricted MIN POWER ratio will follow the same labelling method as the lower bound proof in [2]. We use the observation in Section 2 that the edge tree B_T can be assumed to be a complete binary tree without loss of generality. If $k = 2^r + s (0 \leqslant s < 2^r)$, then by labelling the nodes of B_T, we can construct $r2^r + s$ different $(k-1)$-restricted vertex decompositions of B_T, which are equivalent to so many k-restricted decompositions of T. Next we can show that one of these decompositions gives us the upper bound.

We first make some illustrations for the labelling:

1. Every node in B_T is labelled with a set of size exactly 2^r chosen from the numbers $\{1, 2, \ldots, r2^r + s\}$.

2. The labelling of nodes is determined inductively by the labelling of its r immediate ancestors.

We will use the labelling procedure in [2] as follows:

1. Initiating step:

The node on the 1st level (root) is labelled with the set $\{1, 2, \ldots, 2^r\}$; the nodes on the second level is labelled with the set $\{2^r + 1, 2^r + 2, \ldots, 2 \cdot 2^r\}$; and in general, all the nodes on the ith level, for $1 \leqslant i \leqslant$ r, is labelled with the set $\{(i-1)2^r+1, (i-1)2^r+2, \ldots, i \cdot 2^r\}$.

2. Inductive step:

For a node u at the level $i + 1 (i \geqslant r)$, we shall apply two rules to it.

Rule 1. Find its ancestor v at the level $i + 1 - r$. Suppose v is labelled with a set $S_v = \{l_1, l_2 \ldots, l_{2^r}\}$ and u is v's jth descendant on level $i + 1$, then u is labelled with a set $S_j = \{l_j, l_{j+1} \ldots, l_{2^r+j-s-1}\}$ where we reduce the subscripts $(\bmod 2^r)$ so that they are in the range 1–2^r.

Rule 2. Add to the label set of u those s labels that are not in the label sets of any of its immediate r ancestors.

We can easily see that the following *disjoint property* holds along the inductive procedure:

The label sets of up to r consecutive nodes on a path up the tree are disjoint.

Theorem 5. *For every $k = 2^r + s$ and every tree T, there exists a k-restricted decomposition Q of T satisfying*

$$P(Q) \leqslant \frac{(1+r)2^r + s}{r2^r + s} P(T)$$

which implies

$$\rho_k \leqslant \frac{(1+r)2^r + s}{r2^r + s}.$$

Proof. First, we utilize the labelling procedure to construct $r2^r + s$ different $(k-1)$-restricted decompositions of B_T: for any symbol x in the labelling set, using nodes labelled with x as roots of subtrees (the root of B_T is always used as a root), we obtain a decomposition of B_T. What we want to prove is that every subtree in this decomposition has at most $k-1$ nodes. Without loss of generality, we take the symbol to be 1. First it is obvious that if the root e_1 of B_T is not labelled 1, then the size of the subtree rooted at e_1 is at most $k-1$ (directly obtained from the labelling procedure). Next, we just need to prove that the component rooted at the node v labelled 1 has size at most $k-1$. Obviously, v generates downward a complete binary tree, which is denoted by B_v. Due to the disjoint property we claim that 1 is not in the labelled set of the nodes on the first r levels of B_v. And by Rule 1, at the $r+1$ level, exactly s nodes is labelled with a set not containing 1. Moreover, by Rule 2, children of these nodes must be labelled with a set containing 1. So when we use all nodes containing 1 as roots, the size of B_v could not exceed $2^r - 1 + s = k - 1$.

Fig. 2 shows a labelling and its induced 5-restricted decomposition.

With this observation, we get $r2^r + s$ different $(k-1)$-restricted decompositions of B_T and meanwhile every node in B_T is chosen exactly 2^r times as roots (except the root of B_T which is chosen $r2^r + s$ times as root). According to Eq. (1), the total power-cost contribution of these $r2^r + s$ decompositions is exactly

$$\sum_{i=1}^{r2^r+s} I(R_i) = 2^r(C(T) - c(e_1)),$$

where e_1 is the root of B_T.

Hence we can conclude that there exists one $(k-1)$-restricted vertex decomposition $\hat{R}$ of B_T with its contribution satisfying

$$I(\hat{R}) \leqslant \frac{2^r(C(T) - c(e_1))}{r2^r + s} \leqslant \frac{2^r}{r2^r + s} C(T).$$

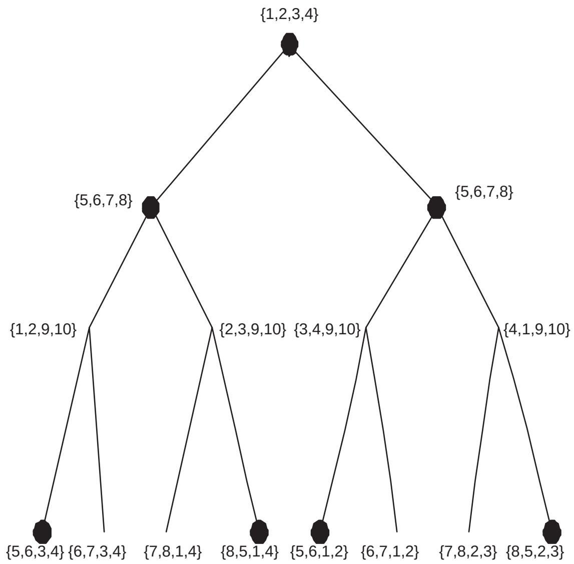

Fig. 2. The labelling procedure and the induced $(k-1)$ restricted decomposition.

If $\hat{Q}$ is the k-restricted decomposition of T corresponding to $\hat{R}$, then by Proposition 4,

$$P(\hat{Q}) \leqslant P(T) + I(\hat{R}) \leqslant P(T) + \frac{2^r}{r2^r + s} C(T).$$

Apply $P(T) \geqslant C(T)$ to the above inequality leads to

$$P(\hat{Q}) \leqslant \frac{(r+1)2^r + s}{r2^r + s} P(T).$$

This completes the proof. □

4. Lower bound for ρ_k

Still assume that $k = 2^r + s$ $(0 \leqslant s < 2^r)$. To prove the lower bound for the k-restricted MIN POWER ratio, we consider a complete binary edge-tree B^n of height n whose nodes at the ith level $(1 \leqslant i \leqslant n+1)$ have weights 2^{n+1-i}. The corresponding tree is denoted by T^n. For a $(k-1)$-restricted vertex decomposition $R = \{B_1, B_2, \ldots, B_p\}$ of B^n and its corresponding k-restricted decomposition Q of T^n, the weight assignment scheme and Proposition 4 ensures that

$$C(T^n) = (n+1)2^n, \tag{5}$$

$$P(T^n) = (n+2)2^n, \tag{6}$$

$$P(Q) = P(T) + I(R). \tag{7}$$

So the problem of estimating $P(Q)$ can be reduced to the problem of estimating $I(R)$.

Before the proof, two terminologies need to be introduced. A subtree B_e of B^n is called a $(k-1)$-*restricted subtree* if the number of its vertices is at most $(k-1)$. B_e is called an *inner subtree* if there is no leaves of B^n in B_e.

Lemma 6. *For any subtree B_e of B^n with root e, the contribution $I(B_e)$ satisfies*

$$I(B_e) \leqslant c(e)$$

and the equality holds if and only if B_e is an inner subtree.

Proof. We construct a subtree $\hat{B}_e$ from B_e by adding all nodes adjacent to B_e in B_n. From the definition of $I(B_e)$, the contribution comes from the weights of all newly added nodes $\hat{B}_e - B_e$. The weight assignment scheme ensures that $I(B_e) \leqslant c(e)$ and the equality holds if and only if all the leaves of B_e are situated above the $(n+1)$th level. □

The next lemma gives the upper bound of $C(B_e)$:

Lemma 7. *For any $(k-1)$-restricted subtree B_e with root e, we have*

$$C(B_e) \leqslant \frac{r2^r + s}{2^r}\, c(e).$$

Proof. We construct a e-rooted $(k-1)$-restricted binary subtree of B^n as $B_{m,e}$ and claim that it maximizes $C(B_e)$ over all e-rooted $(k-1)$-restricted subtrees B_e in B^n. This subtree $B_{m,e}$ covers all the nodes at the r highest levels down from e and s nodes at the $r+1$ level. Since $B_{m,e}$ covers the $k-1$ heaviest nodes of B^n, it must be the subtree with the maximum weight. Then we have $C(B_e) \leqslant C(B_{m,e}) = [(r2^r + s)/2^r]c(e)$. □

Combining Lemma 6, 7 and the fact that $2^r/(r2^r + s)$ decreases monotonously with respect to k, we can draw a corollary immediately:

Corollary 8. *For any $(k-1)$-restricted* INNER *subtree B_e of B^n, we have*

$$I(B_e) \geqslant \frac{2^r}{r2^r + s}\, C(B_e).$$

Now we can arrive at the lower bound theorem:

Theorem 9. *For any $\varepsilon > 0$, there exists a tree T such that for any k-restricted decomposition Q, the power-cost of Q satisfies*

$$P(Q) \geqslant \left(\frac{(1+r)2^r + s}{r2^r + s} - \varepsilon \right) P(T)$$

which implies

$$\rho_k \geqslant \frac{(1+r)2^r+s}{r2^r+s}.$$

Proof. Choose an integer $n>2k$. Let Q be a k-restricted decomposition of T^n, and the corresponding $(k-1)$-restricted decomposition of B^n is $R=\{B_1,B_2,\ldots,B_p\}$. The assumption $n>2k$ ensures that there exists at least one inner subtree. Assume that $\{B_1,B_2,\ldots,B_m\}$ $(m<p)$ are inner subtrees. By Corollary 8, we have

$$I(B_i) \geqslant \frac{2^r}{r2^r+s} C(B_i)(1 \leqslant i \leqslant m).$$

Notice that if B_i situates at the level above $n-k$, it must be an inner tree, hence we have

$$\begin{aligned}
P(Q) &= P(T^n) + \sum_{i=1}^{p} I(B_i) \\
&\geqslant P(T^n) + \sum_{i=1}^{m} I(B_i) \\
&\geqslant P(T^n) + \frac{2^r}{r2^r+s} \sum_{i=1}^{m} C(B_i) \\
&\geqslant P(T^n) + \frac{2^r}{r2^r+s} (n-k)2^n \\
&= P(T^n)\left(1+\frac{2^r}{r2^r+s}\cdot\frac{n-k}{n+2}\right) \\
&= P(T^n)\left(\frac{(1+r)2^r+s}{r2^r+s} - \frac{2^r}{r2^r+s}\cdot\frac{k+2}{n+2}\right).
\end{aligned}$$

The last inequality comes from the fact that every node at the 1st to $(n–k)$th level always belong to some inner subtree. Let n be large enough and the proof completes. □

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